Chart:

t^n $n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$, $s > 0$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$

1. Compute

$$F(s) = \mathscr{L}\{2t^2 + 5e^{2t}\sin(2t)\}$$

$$= 2\mathscr{L}\{t^2\} + 5\mathscr{L}\{e^{2t}\sin(2t)\}$$

$$= 2\left(\frac{2!}{s^3}\right) + 5\left(\frac{2}{(s-2)^2+4}\right)$$

$$= \frac{4}{s^3} + \frac{10}{(s-2)^2+4}$$

(a) $F(s) = \frac{4}{s^3} + \frac{10}{(s-2)^2+4}$

(b) $F(s) = \frac{4}{s^3} + \frac{10}{(s-2)^2+2}$

(c) $F(s) = \frac{2}{s^2} + \frac{10}{(s-2)^2+4}$

(d) None of the above.

2. Which of the below is an appropriate form for the partial fraction decomposition of

$$F(s) = \frac{s^3 + 2s^2 + s - 1}{(s-1)^2[(s+1)^2+4]}$$

(a) $F(s) = \frac{As}{(s-1)} + \frac{Bs+C}{(s+1)^2+4}$

(b) $F(s) = \frac{A}{(s-1)} + \frac{B(s-1)+C}{(s+1)^2+4}$

(c) $F(s) = \frac{A}{(s-1)} + \frac{Bs+C}{(s+1)^2+4}$

(d) None of the above.

Correct form:

$$F(s) = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C(s+1)+2D}{(s+1)^2+4}$$

3. Which of the following is incorrect?

(a) $\mathscr{L}\{cf\} = c\mathscr{L}\{f\}$ for any constant c.

(b) $\mathscr{L}\{f'\} = s\mathscr{L}\{f\}$

(c) $\mathscr{L}\{e^{at}f(t)\} = \mathscr{L}\{f\}(s-a)$

(d) More than one of the above.

Memorized formula:

$$\boxed{\mathscr{L}\{f'\} = s\mathscr{L}\{f\} - f(0)}$$

4. Compute

$$F(s) = \mathscr{L}\{t^4 - 2t^2 - 2\sin(\sqrt{2}t)\}$$

Chart:

t^n $n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}$, $s > 0$

(a) $F(s) = \frac{24}{s^5} + \frac{2}{s^3} + \frac{2}{s^2+4}$

(b) $F(s) = \frac{24}{s^4} + \frac{2}{s^2} + \frac{2\sqrt{2}}{s^2+2}$

(c) $F(s) = \frac{24}{s^5} + \frac{4}{s^3} + \frac{2\sqrt{2}}{s^2+2}$

(d) None of the above.

$$F(s) = \mathscr{L}\{t^4\} - 2\mathscr{L}\{t^2\} - 2\mathscr{L}\{\sin(\sqrt{2}t)\}$$

$$= \frac{4!}{s^5} - 2\left(\frac{2!}{s^3}\right) - 2\left(\frac{\sqrt{2}}{s^2+2}\right)$$

$$= \frac{24}{s^5} - \frac{4}{s^3} - \frac{2\sqrt{2}}{s^2+2}$$

Note: The intended answer was (c) which is incorrect due to a sign error. Answer (d) is therefore correct.

Chart:

$e^{at}t^n \; n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$\cos bt$	$\frac{s}{s^2+b^2}, \quad s > 0$

5. Compute,

$$f(t) = \mathscr{L}^{-1}\left\{\frac{6}{(s-1)^4} + \frac{2s}{s^2+4}\right\}$$

(a) $f(t) = t^3 e^{-t} + \cos(2t)$

(b) $f(t) = t^2 e^{t} + 2\cos(t)$

(c) $f(t) = t^3 e^{t} + 2\cos(2t)$

(d) None of the above.

$$= \mathcal{L}^{-1}\left\{\frac{6}{(s-1)^4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$= t^3 e^t + 2\cos(2t)$$

6. Use the following trigonometric identity

$$\sin(2x) = 2\sin(x)\cos(x)$$

$\Rightarrow \frac{1}{2}\sin(2x) = \sin(x)\cos(x)$

to compute the following

$$F(s) = \mathscr{L}\{\cos(2t)\sin(2t)\}$$

let $x = 2t$

$\frac{1}{2}\sin(4t) = \sin(2t)\cos(2t)$

Chart:

$\sin bt$	$\frac{b}{s^2+b^2}, \quad s > 0$

(a) $F(s) = \dfrac{4}{s^2+16}$

(b) $F(s) = \dfrac{2}{s^2+16}$

(c) $F(s) = \dfrac{2}{s^2+4}$

(d) None of the above.

So, $\mathcal{L}\{\sin(2t)\cos(2t)\}$

$= \frac{1}{2}\mathcal{L}\{\sin(4t)\} = \dfrac{2}{s^2+16}$

7. Using the method of undetermined coefficients, what is an appropriate form for the particular solution to

$$y'' + 2y' - 3y = 2te^t\sin(t)$$

Solution:
See other page

(a) $y_p(t) = (A_1 t + A_0)e^t\cos(t) + (B_1 t + B_0)e^t\sin(t)$

(b) $y_p(t) = (A_1 t)e^t\cos(t) + (B_1 t)e^t\sin(t)$

(c) $y_p(t) = (A_1 t + A_0)e^t\cos(t)$

(d) None of the above.

8. Find the solution to the following differential equation.

$$y'' + y = 2e^t, \quad y(0) = 1, \quad y'(0) = 0$$

(a) $y(t) = e^t - \sin(t)$

(b) $y(t) = e^t - \cos(t)$

(c) $y(t) = e^{-t} + \sin(t)$

(d) None of the above.

Solution:
See other page

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From memory:

To find a particular solution to the differential equation:

$$ay'' + by' + cy = P_m(t)e^{\alpha t}\cos(\beta t) + Q_n(t)e^{\alpha t}\sin(\beta t)$$

where $P_m(t)$ and $Q_n(t)$ are polynomials of degreem m and n respectively, use

$$y_p(t) = t^s(A_k t^k + \ldots + A_1 t + A_0)e^{\alpha t}\cos(\beta t) + t^s(B_k t^k + \ldots + B_1 t + B_0)e^{\alpha t}\sin(\beta t)$$

where $k = \max\{m, n\}$ and with

i) $s = 0$ if $\alpha + i\beta$ is not a root of the auxilliary equation.

ii) $s = 1$ if $\alpha + i\beta$ is a root of the auxilliary equation.

$$y'' + 2y' - 3y = 2te^t\sin(t)$$

Aux Eqn: $r^2 + 2r - 3 = (r+3)(r-1)$

roots: $r = -3, 1$

$P_m(t) = 0$, $Q_n(t) = 2t$, $K = \max\{0, 1\} = 1$

$\alpha = 1, \beta = 1$, $\alpha + i\beta = 1 + i$ Not a root $\Rightarrow s = 0$

$\Rightarrow$ (a) $y_p(t) = (A_1 t + A_0)e^t\cos(t) + (B_1 t + B_0)e^t\sin(t)$

#8

Find the solution to the following differential equation.

$$y'' + y = 2e^t, \quad y(0) = 1, \quad y'(0) = 0$$

① **Laplace Transform:**

$\mathcal{L}\{y'' + y\} = \mathcal{L}\{2e^t\}$

$[s^2 Y - s(1) - 0] + Y = 2\left(\frac{1}{s-1}\right)$

$(s^2+1)Y - s = \frac{2}{s-1}$

$(s^2+1)Y = \frac{2}{s-1} + s$

$Y = \frac{2+s(s-1)}{(s-1)(s^2+1)} = \frac{A}{(s-1)} + \frac{Bs+C}{s^2+1}$

② **Find Coefficients:**

$\Rightarrow \quad 2 + s(s-1) = A(s^2+1) + Bs(s-1) + C(s-1)$

Plug in $s=1$: $2 + s(0) = A(1+1) + B(0) + C(0) \Rightarrow A=1$

plug in $s=0$: $2 + 0(-1) = 1(0+1) + B(0) + C(-1) \Rightarrow C=-1$

plug in $s=2$: $2 + 2(1) = 1(5) + B(2)(1) + (-1)(1) \Rightarrow B=0$

③ **Inverse Laplace Transform:**

Since $A=1,\ B=0,\ C=-1$

$Y = \frac{1}{s-1} - \frac{1}{s^2+1}$

$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s^2+1}\right\}$

$y(t) = e^t - \sin(t)$

chart:

$\sin bt$	$\frac{b}{s^2+b^2}, \quad s > 0$

e^{at}	$\frac{1}{s-a}, \quad s > a$

memory:

$\mathcal{L}\{f''\} = \mathcal{L}\{f\} - sf(0) - f'(0)$

Chart:

e^{at}	$\frac{1}{s-a}$, $s > a$

9. Compute,

$$f(t) = \mathscr{L}^{-1}\left\{\frac{3s-1}{(s-1)(s+1)}\right\}$$

(a) $f(t) = e^t + 2e^t$

(b) $f(t) = e^{-t} + 2e^{-t}$

(c) $f(t) = e^t + 2e^{-t}$

(d) None of the above.

$$\frac{3s-1}{(s-1)(s+1)} = \frac{A}{(s-1)} + \frac{B}{(s+1)}$$

$\Rightarrow 3s-1 = A(s+1) + B(s-1)$

plug in $s=1$: $2 = A(2) \Rightarrow A = 1$

plug in $s=-1$: $-4 = B(-2) \Rightarrow B = 2$

$$\mathcal{L}^{-1}\left\{\frac{3s-1}{(s-1)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^t + 2e^{-t}$$

10. Determine the Laplace transform of the following function:

$$f(t) = t\cos(t)$$

(a) $F(s) = \dfrac{s^2-1}{(s^2+1)^2}$

(b) $F(s) = \dfrac{1-s^2}{(s^2+1)^2}$

(c) $F(s) = \dfrac{1}{s^2+1}$

(d) None of the above.

Chart:

$\cos bt$	$\frac{s}{s^2+b^2}$, $s > 0$

memory:

$$\boxed{\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f\}(s)}$$

$$\mathcal{L}\{t^1\cos(t)\} = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2+1}\right) = -\left(\frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2}\right) = \frac{s^2-1}{(s^2+1)^2}$$

11. Listed below is the Laplace transform for *hyperbolic sine*,

$$\mathscr{L}\{\sinh(\alpha t)\} = \frac{\alpha}{s^2-\alpha^2}.$$

Use the above formula to determine the Laplace transform of the following function:

$$f(t) = e^{4t}\sinh(2t)$$

(a) $F(s) = \dfrac{(s-4)}{(s-4)^2-2}$

(b) $F(s) = \dfrac{2}{(s-4)^2-2}$

(c) $F(s) = \dfrac{2}{(s-4)^2-4}$

(d) None of the above.

memory:

$$\boxed{\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f\}(s-a)}$$

$$\mathcal{L}\{e^{4t}\sinh(2t)\} = \mathcal{L}\{\sinh(2t)\}(s-4) = \frac{2}{(s-4)^2-4}$$

12. Find the partial fraction decomposition of

$$F(s) = \frac{s^2+6}{(s+2)(s^2-2s+2)}$$

(a) $F(s) = \frac{1}{s+2} - \frac{2}{(s-1)^2+1}$

(b) $F(s) = \frac{1}{s+2} + \frac{2}{(s-1)^2+1}$

(c) $F(s) = \frac{1}{s+2} + \frac{1}{(s-1)^2+1}$

(d) None of the above.

<u>Complete the square:</u>

$s^2-2s+2 \ +1 \ -1$

$= s^2-2s+1 \ + (2-1)$

$= (s-1)^2+1$

$$\frac{s^2+6}{(s+2)(s^2-2s+2)} = \frac{A}{(s+2)} + \frac{B(s-1)+C}{[(s-1)^2+1]}$$

$\Rightarrow \ s^2+6 = A[(s-1)^2+1] + B(s-1)(s+2) + C(s+2)$

plug in $s=-2$: $10 = A(10) \Rightarrow A=1$

plug in $s=1$: $7 = 1[0+1] + B(0) + C(3) \Rightarrow C=2$

plug in $s=0$: $6 = 1[1+1] + B(-1)(2) + 2(2) \Rightarrow B=0$